1 (a) $f(x,y,z,\lambda) = z - \lambda(2x^2 + 3y^2 + z^2 - 12xy + 4xz)$

$\frac{\partial f}{\partial x} = -4\lambda x + 12y\lambda - 4\lambda z \Rightarrow x - 3y + z = 0 \Rightarrow$ $\underline{z = 5x}$

$\frac{\partial f}{\partial y} = -\lambda 6y + 12\lambda x \Rightarrow y - 2x = 0 \Rightarrow \underline{y = 2x}$

$\frac{\partial f}{\partial z} = 1 - 2\lambda z - 4\lambda x \Rightarrow 1 = 2\lambda(z + 2x)$

$\Rightarrow 1 = \cancel{10\lambda}$ $2\lambda(7x)$

$\underline{1 = 14\lambda x}$

$x = \frac{1}{14\lambda} = \frac{1}{\alpha}$

$2x^2 + 3y^2 + z^2 - 12xy + 4xz = 35$

$\Rightarrow \frac{2}{\alpha^2} + \frac{3 \cdot 4}{\alpha^2} + \frac{25}{\alpha^2} - \frac{12 \cdot 2}{\alpha^2} + \frac{4 \cdot 5}{\alpha^2} = 35$

$\Rightarrow 2 + 12 + \cancel{25 - 24} + 20 = 35\alpha^2$

$35 = 35\alpha^2 \Rightarrow x = \pm 1 \quad \underline{\underline{z = \pm 5}} \quad y = \pm 2$

$$2(a)\quad \frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) = \cancel{\frac{\partial F}{\partial x}\frac{dx}{dx}}^{\,0} + \frac{\partial F}{\partial y}\frac{dy}{dx} + \cancel{\frac{\partial F}{\partial y'}\frac{dy'}{dx}} - y''\frac{\partial F}{\partial y} \ldots$$

$$= 0 .$$

(b) $\cancel{s}$ $\displaystyle\int_0^1 \left[(y')^2 + 4y - \lambda y\right] dx$ $\qquad y(0) = y(1) = 0$

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \Rightarrow 4 - \lambda - \frac{d}{dx}\left(2y'\right) = 0$$

$$\Rightarrow 2y'' + \lambda - 4 = 0$$

$$\Rightarrow y'' + \frac{\lambda}{2} - 2 = 0$$

$$\Rightarrow y'' = \left(2 - \frac{\lambda}{2}\right)$$

$$\Rightarrow y^{*} = \frac{1}{2}\left(2 - \frac{\lambda}{2}\right)x^2 + Ax + B$$

$\frac{1}{6} - \frac{1}{4}$

$\frac{2}{12} - \frac{3}{12} = -\frac{1}{12}$

$y(0) = 0 : \quad 0 = B .$

$y(1) = 0 \quad 0 = \frac{1}{2}\left(2 - \frac{\lambda}{2}\right) + A \quad \Rightarrow \quad A = \frac{1}{2}\left(\frac{\lambda}{2} - 2\right)$

$$y = \frac{1}{2}\left(2 - \frac{\lambda}{2}\right)x^2 + \frac{1}{2}\left(\frac{\lambda}{2} - 2\right)x$$

$$\int_0^1 y\,dx = \left[\frac{1}{6}\left(2 - \frac{\lambda}{2}\right)x^3 + \frac{1}{4}\left(\frac{\lambda}{2} - 2\right)x^2\right]_0^1 = \frac{1}{6}\left(2 - \frac{\lambda}{2}\right) - \frac{1}{4}\left(2 - \frac{\lambda}{2}\right) \quad := 4$$

$= \cancel{\frac{1}{3} - \lambda}$

$\cdot\ 2\left(2 - \frac{\lambda}{2}\right) - 3\left(2 - \frac{\lambda}{2}\right) = 48$

$4 - \lambda - 6 + \frac{3\lambda}{2} = 48$

$-\lambda + \frac{3\lambda}{2} = 50$

$\cancel{-\frac{1}{12}\left(2 - \frac{\lambda}{2}\right) = 4}$

$\cancel{-\frac{1}{6}(\quad) = 4}$

$\cancel{\rightarrow y = \pm}$

96
$+48$
144

96
$+24$
120

$\frac{\cancel{3}\lambda}{2} = 50 \Rightarrow \lambda = 100$

96
-48

$3\overline{)96}$ 32

$$\Rightarrow y = \frac{1}{2}(2 - 50)x^2 + \frac{1}{2}(48)x$$

$$y = -24x^2 + 24x$$

$$\Rightarrow \int_0^1 (y')^2 + 4y\,dx = \int_0^1 [-48x + 24]^2 - 96x^2 + 96x)\,dx$$

$= \cancel{\int_0^1 \ldots x^2 - 96x^2 + 48x + 24\,dx = \left[-32x^3 + 84x^2 + 24x\right]}$

$$\int_0^1 \left([-48x+24]^2 - 96x^2 + 96x\right) dx$$

$$= \int_0^1 \left(2304x^2 - 2304x + 576 - 96x^2 - 96x\right)$$

$$\begin{array}{r} 48 \\ \times\ 48 \\ \hline \end{array} \qquad \begin{array}{r} 2400 \\ -\ 96 \\ \hline 2304 \end{array}$$

$$= \int_0^1 2208x^2 - 2208x + 576 \, dx$$

$$= \left[736x^3 - 1104x^2 + 576x\right]_0^1 = \underline{208}.$$

3(b) contd.

$$\frac{du}{dx} + \frac{1-\phi}{1+\phi}\left(\frac{u}{x}\right) = -2(1-\phi).$$

$$\text{I.F.} = e^{\int \frac{1-\phi}{1+\phi} \cdot \frac{1}{x} dx} = e^{\frac{1-\phi}{1+\phi} \ln x} = x^{\frac{1-\phi}{1+\phi}}$$

$$\rightarrow \frac{d}{dx}\left(u x^{\frac{1-\phi}{1+\phi}}\right) = -2(1-\phi) x^{\frac{1-\phi}{1+\phi}}$$

$$\rightarrow u x^{\frac{1-\phi}{1+\phi}} = -2(1-\phi) x^{\frac{2}{1+\phi}} \cdot \frac{(1+\phi)}{2} + C(\phi)$$

$$u x^{\frac{1-\phi}{1+\phi}} = -\cancel{2}(1-\phi^2) x^{\frac{2}{1+\phi}} + C(\phi)$$

$$\rightarrow u = -(1-\phi^2) x + x^{-\frac{1-\phi}{1+\phi}} C(\phi)$$

$$\cancel{\frac{1+\frac{y}{x}}{1-\frac{y}{x}}} \quad \cancel{\frac{x+y}{x-y}}$$

$$\frac{1-\frac{y}{x}}{1+\frac{y}{x}} \quad \frac{x-y}{x+y}$$

$$\phi = \frac{y}{x}: \quad u = \left(\frac{y^2}{x^2} - 1\right) x + x^{\frac{y-x}{x+y}} c\left(\frac{y}{x}\right)$$

3 (a) $\frac{\partial u}{\partial y} + u\frac{\partial u}{\partial x} = -\lambda u$

$\underline{t=0}$
$x=s$
$y=0$
$u=f(s)$

$\frac{dx}{dt}=u$ $\quad$ $\frac{dy}{dt}=1$ $\quad$ $\frac{du}{dt}=-\lambda u$

$\ln u = -\lambda t + c$

i.e. $u = e^{-\lambda t}K$

$y = t + y_0$

$\Rightarrow \underline{y=t}$

i.e. $\underline{u = e^{-\lambda t} f(s)}$

$\Rightarrow f(s) = ue^{\lambda t}$

$\frac{dx}{dt} = e^{-\lambda t} f(s)$

$x = -\frac{1}{\lambda}e^{-\lambda t} f(s) + d$

$\underline{x = \frac{f(s)}{\lambda} + s - \frac{1}{\lambda}e^{-\lambda t} f(s)}$

$x = f(s)\left[\frac{1}{\lambda} - \frac{1}{\lambda}e^{-\lambda t}\right] + s = \underline{f(s)\frac{1}{\lambda}(1-e^{-\lambda t}] + s}$

$\Rightarrow f(s) = x - s$

$x = \frac{ue^{\lambda t}}{\lambda}[1-e^{-\lambda t}] + s$

$\Rightarrow s = x - \frac{ue^{\lambda t}}{\lambda}(1-e^{-\lambda t})$

$\Rightarrow u = e^{-\lambda t} f\left(x - \frac{u}{\lambda}(e^{\lambda t}-1)\right).$

(b) $$x(x+y)\frac{\partial u}{\partial x} + y(x+y)\frac{\partial u}{\partial y} = -(x-y)(2x+2y+u)$$

$$\phi(x,y) = \frac{y}{x} \qquad \frac{d\phi}{dt} = \frac{\partial \phi}{\partial x}\frac{dx}{dt} + \frac{\partial \phi}{\partial y}\frac{dy}{dt}$$

$$= -\frac{y}{x^2}\dot{x} + \frac{1}{x}\dot{y}$$

Chs: $\frac{dx}{dt} = x(x+y)$

$\frac{dy}{dt} = y(x+y)$ $\longrightarrow$ $= -\frac{y}{x^2}x(x+y) + \frac{1}{x}y(x+y) = \underline{\underline{0}}.$

Let $\phi = \frac{y}{x}$ then $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \phi}\frac{\partial \phi}{\partial x} + \frac{\partial u}{\partial \psi}\frac{\partial \psi}{\partial x} = -\frac{\partial u}{\partial \phi}\frac{y}{x^2} + \frac{\partial u}{\partial \psi}$

Let $\psi = x$. then $\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \phi}\frac{\partial \phi}{\partial y} + \frac{\partial u}{\partial \psi}\frac{\partial \psi}{\partial y} = \frac{\partial u}{\partial \phi}\frac{1}{x}$

$$\Rightarrow x(x+y)\left[-\frac{\partial u}{\partial \phi}\frac{y}{x^2} + \frac{\partial u}{\partial \psi}\right] + y(x+y)\frac{\partial u}{\partial \phi}\frac{1}{x} = -(x-y)(2x+2y+u)$$

$$\Rightarrow \cancel{-\frac{y(x+y)}{x}\frac{\partial u}{\partial \phi}} + x(x+y)\frac{\partial u}{\partial \psi} + \cancel{\frac{y}{x}(x+y)\frac{\partial u}{\partial \phi}} = -(x-y)(2x+2y+u)$$

ie. $$x(x+y)\frac{\partial u}{\partial \psi} = -(x-y)(2x+2y+u)$$

ie. $$\psi^2(1+\phi)\frac{\partial u}{\partial \psi} = -\psi(1-\phi)(2\psi + 2\psi\phi + u)$$

$$\frac{\partial u}{\partial \psi} = -\frac{(1-\phi)}{(1+\phi)}\frac{(2\psi + 2\psi\phi + u)}{\psi}$$

$$= -\frac{(1-\phi)}{(1+\phi)}(2 + 2\phi + u/\psi)$$

$$\frac{\partial u}{\partial \psi} = -2(1-\phi) - \frac{(1-\phi)}{(1+\phi)}\frac{u}{\psi}$$

ie. $$\frac{\partial u}{\partial \psi} + \frac{(1-\phi)}{(1+\phi)\psi}u = -2(1-\phi)$$

4. look for sol^{n}s of form $z=f(x+mt) \Rightarrow f'' = \frac{m^2}{c^2} f'' \Rightarrow m$

$\Rightarrow z = f(x+ct) + g(x-ct)$.

$t=0$: $F(x) = f(x) + g(x)$

$G(x) = cf'(x) - cg'(x)$

$\Rightarrow \int_\alpha^x G(s)\,ds = cf(x) - cg(x)$

$\Rightarrow f(x) = F(x) + \frac{1}{c}\int_\alpha^x G(s)\,ds$

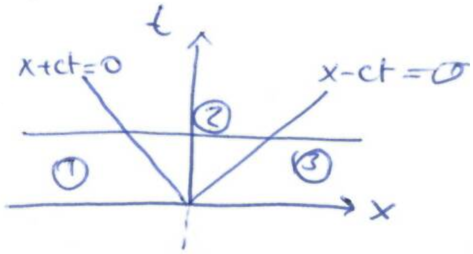
$\Rightarrow 2f(x+ct) = F(x+ct) + \frac{1}{c}\int_\alpha^{x+ct} G(s)\,ds$

$\Rightarrow 2g(x-ct) = F(x-ct) - \frac{1}{c}\int_\alpha^{x-ct} G(s)\,ds$

$$\Rightarrow z = \frac{1}{2}\left[F(x+ct) + F(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)\,ds.$$

$F(x) = 0$

$$G(x) = \begin{cases} \frac{1}{1+x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$



$$z(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)\,ds$$

In ①, $z(x,t) = 0$

②, $z(x,t) = \frac{1}{2c}\int_0^{x+ct} G(s)\,ds = \frac{1}{2c}\int_0^{x+ct} \frac{1}{1+s}\,ds = \frac{1}{2c}\left[\ln(1+s)\right]_0^{x+ct}$

$= \frac{1}{2c}\ln(1+x+ct)$

③, $z(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)\,ds = \frac{1}{2c}\int_{x-ct}^{x+ct}\frac{1}{1+s} = \frac{1}{2c}\ln(1+x+ct) - \frac{1}{2c}\ln(1+x-ct)$

z is continuous: on $x+ct=0 \Rightarrow$ ① $=0$, ② $=0$ ✓

on $x-ct=0 \Rightarrow$ ② $= \frac{1}{2c}$ ③ ✓

and as $|x| \to 0$

① $= 0$

② $= \frac{1}{2c}\ln(1+ct)$

③ $= \frac{1}{2c}\ln(1+ct) - \frac{1}{2c}\ln(1-ct)$

$z_t =$ ① 0

② $\cancel{\frac{1}{2(1+x+ct)}}$ $\cancel{t=0}$

③ $\frac{1}{2(1+x+ct)} + \frac{1}{2(1+x-ct)}$ at $t=0$ $= G$ ✓.

5. (i) $\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial \theta}{\partial t} = 0$ for steady sol^{n}s.

$$\Rightarrow \theta = Ax + B + f(t)$$
$$\theta = C \qquad \Rightarrow \qquad \theta = Ax + B$$

$0 = -LA + B \Rightarrow B = LA$
$T = LA + LA \Rightarrow A = \frac{T}{2L}$

$$\theta = \frac{Tx}{2L} + \frac{LT}{2L}$$
$$\theta_0 = \frac{T}{2}\left(1 + \frac{x}{L}\right)$$

(ii) $\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial \theta}{\partial t} = 0$ for steady sol^{n}s.

$\Rightarrow \theta = Ax + B$

$$T = A(-L) + B$$
$$0 = AL + B$$
$$\Downarrow$$
$$B = \frac{T}{2}$$
$$A = -\frac{T}{2L}$$

$$\theta_s = -\frac{T}{2L}x + \frac{T}{2} = \frac{T}{2}\left(1 - \frac{x}{L}\right)$$

Write $\theta = \theta_s + \theta_u(x,t) \Rightarrow \frac{d^2\theta_u}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial \theta_u}{\partial t}$

$\theta_u(L,t) = 0$
$\theta_u(L,t) = 0.$
$\theta_u(x,0) = \theta_0 - \theta_s$
$= \frac{Tx}{L}$

$\theta_u = X(x)\mathcal{T}(t)$

$$X''T = \frac{1}{\alpha^2}XT'$$
$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T} = -p^2$$

$$X'' + p^2X = 0 \Rightarrow X = A\cos px + B\sin px$$

b.c. $0 = A\cos p(-L) + B\sin p(-L)$
$0 = A\cos pL + B\sin pL \qquad \Rightarrow A = 0$

$\Rightarrow pL = n\pi \Rightarrow p = \frac{n\pi}{L} \quad \to \quad X = B_n \sin\frac{n\pi x}{L}$

$$T' + p^2\alpha^2 T = 0 \Rightarrow T = e^{-p^2\alpha^2 t} = e^{-n^2\pi^2\alpha^2 t/L^2}$$

$$\theta_u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\, e^{-n^2\pi^2\alpha^2 t/L^2}$$

$$\Rightarrow \quad \Theta(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} e^{-\alpha^2 n^2 \pi^2 t/L} + \Theta_s$$

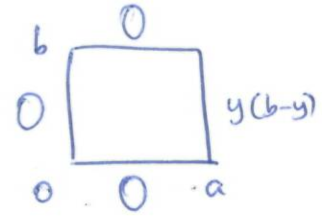
~~At t=0~~ $\cancel{= \Theta(0,t) - \Theta_s = \sum_{n=1}^{\infty} B_n}$

B_n chosen s.t.

At $t=0$

$$\Theta_0 - \Theta_s = \frac{Tx}{L} = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \quad \Rightarrow$$

6.



$$u = X(x)Y(y)$$

$$X''Y + Y''X = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = p^2$$

$$\cancel{X = A\cos px + B\sin px}$$

$$\cancel{Y = C\cosh py + D\sinh py}$$

~~b.c. 0~~

$$X = A\cosh px + B\sinh px$$

$$Y = C\cos py + D\sin py$$

$$0 = C \rightarrow Y = D\sin py$$

$$0 = D\sin pb \Rightarrow pb = n\pi \Rightarrow p = \frac{n\pi}{b}$$

$$0 = A \Rightarrow X = B\sinh\left(\frac{n\pi}{b}x\right)$$

$$\cancel{y(b-y) = B\sinh\left(\frac{n\pi}{b}a\right)}$$

$$\Rightarrow u = \sum_{n=1}^{\infty} B_n \sinh\left(\frac{n\pi x}{b}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$yb - y^2$

$b - 2y$

Then $$y(b-y) = \sum_{n=1}^{\infty} B_n \sinh\left(\frac{n\pi a}{b}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$\int_0^b y(b-y)\cdot\sin\left(\frac{m\pi y}{b}\right)dy = \frac{1}{2}b B_n \sinh\left(\frac{n\pi a}{b}\right)$$

$$= \left[y(b-y)\frac{b}{m\pi}\cos\left(\frac{m\pi y}{b}\right)\right]_b^0 + \frac{b}{m\pi}\int (b-2y)\cos\left(\frac{m\pi y}{b}\right)dy$$

$$= \frac{b}{m\pi}\int_0^b (b-2y)\cos\left(\frac{m\pi y}{b}\right)dy = \frac{b}{m\pi}\left[\left[(b-2y)\sin\left(\frac{m\pi y}{b}\right)\frac{b}{m\pi}\right]_0^b + \frac{b}{m\pi}\int_0^b 2\sin\left(\frac{n\pi y}{b}\right)dy\right]$$

$=$